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ON RATIONALLY CONVEX HULLS

BY

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ABSTRACT. For a compact set $X \subseteq \mathbf{C}^n$, let $h_r(X)$ denote the rationally convex hull of X; let Δ denote the closed unit disk in $\mathbf{C}$; and, following Wermer, for a compact set S such that $\partial\Delta \subseteq S \subseteq \Delta$ let $X_S = S \times S \cap \partial\Delta^2$. It is shown that

$$h_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$$

where u_S is a function on S which, in the case when S is smoothly bounded, is specified by requiring $u_S|_{\partial\Delta} = 0$, $u_S|_{\partial S \setminus \partial\Delta} = 1$ and $u_S|_{\operatorname{int} S}$ harmonic. In particular this provides a precise description of $h_r(X)$ for certain sets $X \subseteq \mathbf{C}^2$ with the property that $h_r(X) \neq X$, but $h_r(X)$ does not contain analytic structure (as Wermer demonstrated, there are S for which $X = X_S$ has these properties). Furthermore, it follows that whenever $h_r(X_S) \neq X_S$ then there is a Gleason part of $h_r(X_S)$ for the algebra $R(X_S)$ with positive four-dimensional measure. In fact, the Gleason part of any point $(z, w) \in h_r(X_S) \cap \operatorname{int}\Delta^2$ such that $u_S(z) + u_S(w) < 1$ has positive four-dimensional measure.

A similar idea is then used to construct a compact rationally convex set $Y \subseteq \mathbf{C}^2$ such that each point of Y is a peak point for $R(Y)$ even though $R(Y) \neq C(Y)$; namely, $Y = \tilde{X}_T = \{(z, w) \in \mathbf{C}^2 \mid z \in T, |w| = \sqrt{1 - |z|^2}\}$ where T is any compact subset of $\operatorname{int}\Delta$ having the property that $R(T) \neq C(T)$ even though there are no nontrivial Jensen measures for $R(T)$. This example is more concrete than the original example of such a uniform algebra which was discovered by Cole. It is possible to show, for instance, that $R(\tilde{X}_T)$ is not even in general locally dense in $C(\tilde{X}_T)$, a possibility which had been suggested by Stuart Sidney.

Finally, smooth examples (3-spheres in $\mathbf{C}^6$) with the same pathological properties are obtained from X_S and $\tilde{X}_T$.

Introduction. Let X be a compact subset of $\mathbf{C}^n$. $R_0(X)$ is the algebra of all

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rational functions P/Q on $\mathbf{C}^n$ with P, Q polynomials and $Q \neq 0$ on X. $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$. $h_r(X)$, the rationally convex hull of X, is the set $\{z \in \mathbf{C}^n \mid$ for all polynomials P such that $P(z) = 0$, P has a zero on $X\}$.

While every $X \subseteq \mathbf{C}$ is rationally convex, i.e., $h_r(X) = X$, in $\mathbf{C}^2$ even a compact arc may fail to be rationally convex. (See the example of Wermer [1] and Rudin [2] or [3, p. 70]. Wermer's example is in $\mathbf{C}^3$.) The rationally convex hull of a general "circled" compact set in $\mathbf{C}^n$ has been described very thoroughly by de Leeuw [4], [5] and Rossi [6]. A set $X \subseteq \mathbf{C}^n$ is "circled" if, whenever $(z_1, \cdots, z_n) \in X$, $(\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ and $|\lambda_j| = 1$ for $1 \leq j \leq n$, it follows that $(\lambda_1 z_1, \cdots, \lambda_n z_n) \in X$. Suppose that X is a "circled" compact set in $\mathbf{C}^n$, that $\operatorname{int} X$ is a dense connected subset of X, and that $\operatorname{int} X$ meets every coordinate axis $\{z_j = 0\}$ which X meets. In this case their result is that

$$h_r(X) = \text{closure}\,\{(z_1, \cdots, z_n) \in \mathbf{C}^n \mid (\log|z_1|, \cdots, \log|z_n|)$$
$$\in \text{linear convex hull}\,\{(s_1, \cdots, s_n) \in \mathbf{R}^n \mid (e^{s_1}, \cdots, e^{s_n}) \in X\}\}.$$

The general case follows by taking a decreasing intersection of such sets (see also [3, pp. 71–76]).

In the first section of this paper we describe $h_r(X)$ for certain sets X contained in the boundary of the bicylinder which were first discussed by Wermer [7]. Given a compact set S contained in the closed unit disk Δ and containing the boundary of the disk $\partial\Delta$, let

$$X_S = \{(z, w) \in \mathbf{C}^2 \mid |z| = 1 \quad \text{and} \quad w \in S, \text{ or } |w| = 1 \quad \text{and} \quad z \in S\}.$$

A set $Y \subseteq \mathbf{C}^n$ is said to "contain analytic structure" if there is a nonconstant analytic mapping from a disk to $\mathbf{C}^n$ whose image is contained in Y. As was observed by Soltzenberg [8], the open mapping theorem implies that if Y contains analytic structure, then one of the coordinate projections $z_i(Y)$ has nonempty interior in $\mathbf{C}$. Wermer in [7] constructed an S such that $h_r(X_S) \setminus X_S \neq \emptyset$ but $\operatorname{int} S = \emptyset$, whence $h_r(X_S) \subseteq S \times S$ does not contain analytic structure. (The first example of this general type is due to Stolzenberg [8], who proved the existence of a compact set $X \subseteq \mathbf{C}^2$ which is not polynomially convex but whose polynomially convex hull does not contain analytic structure.)

First assume that S is a compact connected subset of Δ containing $\partial\Delta$, and write $S = \bigcap_{n=1}^{\infty} S_n$ where

(i) each S_n is a compact connected smoothly bounded subset of Δ;

(ii) S_n is a neighborhood of S_{n+1} in Δ.

Define $u_n \in C(S_n)$ by requiring that u_n be harmonic on $\operatorname{int} S_n$, $u_n \equiv 0$ on $\partial\Delta$

and $u_n \equiv 1$ on $\partial S_n \setminus \partial\Delta$. For $z \in S$ let

$$u_S(z) = \lim_{n \to \infty} u_n(z).$$

The main result of §1 may then be stated as

Theorem 1. $h_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$.

The determination of $h_r(X_S)$ yields some quantitative information:

Theorem 2. *If* $(z, w) \in h_r(X_S) \setminus X_S$ *and* $u_S(z) + u_S(w) < 1$, *then the Gleason part of* (z, w) *in* $R(X_S)$ *has positive 4-dimensional measure.*

This leads almost at once to

Corollary. *If* $h_r(X_S) \neq X_S$, *then the 4-dimensional measure of* $h_r(X_S) \setminus X_S$ *is positive.*

One consequence of Theorem 1 is that there are compact connected subsets S of Δ which contain $\partial\Delta$ and for which X_S is rationally convex even though $R(X_S) \neq C(X_S)$. For example, let S be a compact connected subset of Δ such that $S \supseteq \partial\Delta$, $R(S) \neq C(S)$, but the only Jensen measures for $R(S)$ are point masses (such an S can be obtained by a simple modification of an example due to Browder (see [9, p. 192])). It can be shown that $u_S \equiv 1$ on $S \setminus \partial\Delta$, so that $h_r(X_S) = X_S$ by Theorem 1. This suggests that such S might yield interesting rationally convex sets in other ways.

The second section of this paper is essentially independent of the first. In it we exhibit a concrete example of a uniform algebra A defined on a compact metric space (its maximal ideal space) with the properties:

(I) A does not contain all continuous functions on its maximal ideal space;
(II) every point of the maximal ideal space of A is a peak point for A;
(III) A is finitely generated.

The first example of such an algebra is due to Cole (unpublished); his example is a modification of an algebra satisfying (I) and (II) which is constructed in his thesis ([10]; or see [9, pp. 255–262]).

Let T be any compact subset of $\operatorname{int}\Delta$ such that the only Jensen measures for $R(T)$ are trivial, but $R(T) \neq C(T)$; Cole's construction also begins with such a T (examples of such T, besides the previously mentioned one in [9], are found in [11] and [12]). We use T to obtain a compact subset of the boundary of the unit ball in $\mathbf{C}^2$ by defining

$$\widetilde{X}_T = \{(z, w) \in \mathbf{C}^2 |\ |z|^2 + |w|^2 = 1,\ z \in T\}.$$

Theorem 3. *If* $A = R(\widetilde{X}_T)$, *then conditions* (I), (II) *and* (III) *are satisfied.*

In the third section we show that under suitable restrictions on T, the properties of $\widetilde{X}_T$ are enjoyed locally. This does not follow from the general arguments used in the second section, because there the construction of an annihilating measure for $R(\widetilde{X}_T)$ is not local.

Let $T \subseteq \{|z| < 1\}$ be a "Swiss cheese" in a sense to be made precise later. (The previously mentioned example of Browder satisfies our definition.)

Theorem 4. *If* K *is a compact subset of* $\widetilde{X}_T$ *with nonempty interior in the topology of* $\widetilde{X}_T$, *then the restriction of* $R(\widetilde{X}_T)$ *to* K *is not dense in* $C(K)$.

In the fourth and concluding section we demonstrate that no smoothness assumptions alone can be sufficient to avoid the pathological behavior exhibited by the sets X_S and $\widetilde{X}_T$ discussed in the first two sections. Here we will be interested in the algebra $P(X)$ and its maximal ideal space $h(X)$, defined for a compact set $X \subseteq \mathbf{C}^n$ as follows:

$P(X)$ is the uniform closure in $C(X)$ of the polynomials in the coordinate functions $z_1, \cdots, z_n$.

$h(X)$, the polynomially convex hull of X, is the set $\{z \in \mathbf{C}^n |$ for all polynomials p in $z_1, \cdots, z_n$, $|p(z)| \le \max_X |p|\}$.

Starting with the example in [7], or with the alternative use of Wermer's construction on p. 365, we prove the following.

Theorem 5. *There is a* C^∞ *3-sphere* Σ_1 *contained in* $\mathbf{C}^6$ *such that* $h(\Sigma_1)$ *contains points not in* Σ_1 *but does not contain analytic structure.*

We obtain a similar result by starting instead with the set $\widetilde{X}_T$ of the second section.

Theorem 6. *There is a* C^∞ *3-sphere* Σ_2 *contained in* $\mathbf{C}^6$ *such that* $h(\Sigma_2) = \Sigma_2$, *every point of* Σ_2 *is a peak point for* $P(\Sigma_2)$, *but* $P(\Sigma_2) \neq C(\Sigma_2)$. *In other words* $P(\Sigma_2)$ *satisfies the above conditions* (I), (II) *and* (III).

1. We will employ the following notation throughout:

Δ will denote the closed unit disk; Δ^2 is then the closed unit bicylinder $\{(z, w) \in \mathbf{C}^2 |\ |z| \le 1,\ |w| \le 1\}$;

∂X will denote the topological boundary of a set $X \subseteq \mathbf{C}$, $\mathbf{C}^2$ or $\mathbf{C}^n$ depending on the context;

X_T, where $T \subseteq \Delta$, will denote the set $\{(z, w) \in \Delta^2 |\ z \in T$ and $|w| = 1$ or $|z| = 1$ and $w \in T\}$;

S (or S_n) will denote a closed connected subset of Δ such that $\partial\Delta \subseteq S$ (or S_n).

The following lemma, which is stated in a slightly different form in [7], will be useful to us:

Lemma 1. *Fix* $p \in \Delta^2$. *If* $p \notin h_r(X_S)$, *then* $p \in h(K)$ *for some compact set* $K \subseteq X_{\Delta \setminus S}$.

Proof. If $p \notin h_r(X_S)$, then there is some polynomial P such that $P(p) = 0$, but P has no zero on X_S. Let V be the component of $\{x \in \mathbf{C}^2 | \, P(x) = 0\}$ which contains p, and let $K = V \cap \partial\Delta^2$. By assumption, $K \cap X_S = \emptyset$, so that $K \subseteq \{(z, w) \in \partial\Delta^2 | \, z \notin S \text{ or } w \notin S\} = X_{\Delta \setminus S}$. By the maximum principle, if Q is any polynomial on $\mathbf{C}^2$, then $|Q(p)| \leq \sup_{x \in K} |Q(x)|$, so $p \in h(K)$ as claimed.

Definition. Given S as above, it is possible to choose a sequence $\{S_n\}_{n=1}^{\infty}$ of closed connected subsets of Δ which contain $\partial\Delta$ and satisfy

(i) S_n is bounded by finitely many disjoint smooth curves;

(ii) S_n is a neighborhood of S_{n+1} in Δ;

(iii) $S = \bigcap_{n=1}^{\infty} S_n$.

Let u_n be the unique function in $C(S_n)$ which is harmonic on $\operatorname{int} S_n$ and satisfies $u_n | \partial\Delta \equiv 0$, $u_n | \partial S_n \setminus \partial\Delta \equiv 1$. For $z \in S$ we define $u_S(z) = \lim_{n\to\infty} u_n(z)$. By the maximum principle the functions u_n are monotone increasing, so $u_S(z)$ exists for each $z \in S$. For future reference note that $u_S(z) > 0$ on $S \setminus \partial\Delta$ whenever $S \neq \Delta$.

Lemma 2. u_S *is independent of the choice of* $\{S_n\}$.

Proof. Let $\{S'_m\}_{m=1}^{\infty}$ be another sequence of compact sets as above, and let $\{u'_m\}$ be the corresponding functions in $C(S'_m)$.

Given any n, S_n is a neighborhood in Δ of S_{n+1}, hence also of S. Since $\bigcap_{m=1}^{\infty} S'_m = S$, there is an integer m_0 such that S_n is a neighborhood of S'_{m_0}; in particular, $\partial S'_{m_0} \subseteq S_n$, so $u'_{m_0} \geq u_n$ on S'_{m_0}. Thus we have for $z \in S$:

$$\lim_{m\to\infty} u'_m(z) \geq u'_{m_0}(z) \geq u_n(z).$$

Since n was arbitrary, $\lim_{m\to\infty} u'_m(z) \geq \lim_{n\to\infty} u_n(z) = u_S(z)$. Similarly, $\lim_{m\to\infty} u'_m(z) \leq u_S(z)$ for all $z \in S$.

Observe that if S is "nice," then u_S has the following properties.

Lemma 3. *If* S *is bounded by a finite number of disjoint smooth curves, then* $u_S \in C(S)$, u_S *is harmonic on* $\operatorname{int} S$, *and* $u_S \,|\, \partial\Delta \equiv 0$, $u_S \,|\, \partial S \setminus \partial\Delta \equiv 1$.

Proof. Pick $\{S_n\}$ satisfying (i), (ii), (iii) in the definition of u_S, and let $\{u_n\}$ be the corresponding functions used to define u_S. Let $u \in C(S)$ be harmonic on

$\operatorname{int} S$ and satisfy $u \mid \partial\Delta \equiv 0$, $u \mid \partial S \backslash \partial\Delta \equiv 1$. Since S is smoothly bounded, there is a sequence of harmonic functions $\{v_m\}$ each defined in some neighborhood of S, such that v_m converges to u uniformly on S (see for example, [9, p. 184]).

If $z \in S$, then for all n, $u_n(z) \leq u(z)$ by the maximum principle, so $u_S(z) \leq u(z)$. To prove the converse, we may assume that $v_m \leq 0$ on $\partial\Delta$ and $v_m \leq 1 - 1/m$ on S. Given any m, we choose n sufficiently large that $S_n \subseteq \{v_m \leq 1\}$; then $v_m \mid \partial S_n \leq u_n \mid \partial S_n$, so that $v_m(z) \leq u_n(z) \leq u_S(z)$. Since m was arbitrary, $u(z) = \lim_{m\to\infty} v_m(z) \leq u_S(z)$, whence $u_S = u$.

Our main result may now be stated.

Theorem 1. $h_r(X_S) = \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$.

We first prove two lemmas.

Lemma 4. *Let B be the annulus $\{z \in \mathbf{C} \mid 1 \geq |z| \geq r\}$. Then $h_r(X_B) = \{(z, w) \in \Delta^2 \mid |zw| \geq r\}$. (This is a special case of Theorem 1, since by Lemma 3, $u_B = (\log |z|)/(\log r)$. It is also a special case of a similar theorem for "circled" subsets of $\mathbf{C}^n$, referred to in the introduction.)*

Proof. Fix $(z_0, w_0) \in \Delta^2$. If $|z_0 w_0| < r$, the polynomial $P(z, w) = zw - z_0 w_0$ vanishes at (z_0, w_0), but has no zero on X_B, so $(z_0, w_0) \notin h_r(X_B)$.

Conversely, suppose that $|z_0 w_0| \geq r$. The polynomial $P(z, w) = zw$ is larger in modulus at (z_0, w_0) than it is on any compact subset of $X_{\Delta \backslash B}$. By Lemma 1, $(z_0, w_0) \in h_r(X_B)$.

Lemma 5. *Let X be a compact subset of $\mathbf{C}^n$ and let $\Phi: X \to \mathbf{C}^m$. Suppose that $z_i \circ \Phi \in R(X)$ for each coordinate projection z_i, $i = 1, 2, \cdots, m$. Then $\Phi(h_r(X)) \subseteq h_r(\Phi(X))$.*

Proof. Fix $p_0 \in h_r(X)$, and suppose that P is a polynomial on $\mathbf{C}^m$ such that $P(\Phi(p_0)) = 0$. We must show that P has a zero on $\Phi(X)$.

Any function $F \in R(X)$ which has a zero on $h_r(X)$ must have a zero on X. Indeed, suppose $F(p) = 0$ for some $p \in h_r(X)$. Choose $f_n \in R_0(X)$ such that $\sup_X |F - f_n| \leq 1/n$. Let $g_n = f_n - f_n(p)$, then $g_n(p) = 0$, and $g_n = P_n/Q_n$ for some polynomials P_n, Q_n with $Q_n \neq 0$ on X hence on $h_r(X)$. So $P_n(p) = 0$, and since $p \in h_r(X)$, $P_n(p_n) = 0$ for some $p_n \in X$. Then $g_n(p_n) = 0$, so that

$$|F(p_n)| \leq |F(p_n) - f_n(p_n)| + |f_n(p_n) - f_n(p)| + |f_n(p) - F(p)|$$
$$\leq 1/n + |g_n(p_n)| + 1/n = 2/n.$$

Some subsequence $p_{n_k} \to p_\infty \in X$, so that $F(p_\infty) = 0$ for some $p_\infty \in X$ as claimed.

In our case, $P(\Phi) \in R(X)$ and $P(\Phi)(p_0) = 0$, so we conclude that $P(\Phi)(p) = 0$ for some $p \in X$. Thus P has a zero at $\Phi(p) \in \Phi(X)$ as desired.

Corollary. *Let S be finitely connected. Suppose that ϕ is a one to one continuous map from S into Δ such that $\phi(\partial\Delta) = \partial\Delta$ and ϕ is holomorphic on* int S. *Define $\Phi: h_r(X_S) \to \Delta^2$ by $\Phi(z, w) = (\phi(z), \phi(w))$. Then $\Phi(h_r(X_S)) = h_r(X_{\phi(S)})$.*

Proof. $\phi \in R(S)$ since S is finitely connected (see, for example, [9, p. 231] or [3, p. 51]), and similarly $\phi^{-1} \in R(\phi(S))$. Since evidently $X_S \subseteq S \times S$, $h_r(X_S) \subseteq S \times S$, so that $z \circ \Phi$ and $w \circ \Phi$ are in $R(h_r(X_S))$; similarly $z \circ \Phi^{-1}$ and $w \circ \Phi^{-1}$ are in $R(h_r(X_{\phi(S)}))$. Observe that

$$X_{\phi(S)} = \{(\zeta, \eta) |\ |\zeta| = 1,\ \eta \in \phi(S) \text{ or } |\eta| = 1,\ \zeta \in \phi(S)\}$$

$$= \{(\phi(z), \phi(w)) |\ |z| = 1,\ w \in S \text{ or } |w| = 1,\ z \in S\} = \Phi(X_S).$$

Lemma 5 applied to Φ yields

$$\Phi(h_r(X_S)) \subseteq h_r(\Phi(X_S)) = h_r(X_{\phi(S)}),$$

and applying Lemma 5 to Φ^{-1} we have conversely

$$h_r(X_{\phi(S)}) = \Phi(\Phi^{-1}[h_r(X_{\phi(S)})]$$

$$= \Phi(\Phi^{-1}[h_r(\Phi(X_S))]) \subseteq \Phi(h_r(\Phi^{-1} \circ \Phi(X_S))) = \Phi(h_r(X_S)).$$

Proof of Theorem 1. We first prove the theorem for those S of the form $S = \Delta \setminus \bigcup_{i=1}^{n} \operatorname{int} D_i$, where the D_i are disjoint closed disks in int Δ; then we shall deduce the general case by a "limit" argument. To avoid triviality we assume $n > 0$. In the simplest case $n = 1$, we can reduce the problem to the situation of Lemma 4 by mapping S onto an annulus via a linear fractional transformation, and applying the corollary to Lemma 5. For $n > 1$ we would like to imitate this procedure by defining $\phi: S \to \{1 \geq |\zeta| \geq r\}$ such that $\phi(\partial\Delta) = \{|\zeta| = 1\}$ and $\phi(\partial D_i) = \{|\zeta| = r\}$ for $i = 1, 2, \cdots, n$, with ϕ analytic on int S. Although in general one can only construct a ϕ with approximately these properties—and, of course, ϕ will not be 1-to-1—it turns out that we can again reduce the problem to the situation of Lemma 4.

Fix $(a, b) \in S \times S$.

Case 1. Suppose that $u_S(a) + u_S(b) < 1$. We will show that $(a, b) \in h_r(X_S)$ by proving that any polynomial which vanishes at (a, b) also has a zero on X_S. So let $P(a, b) = 0$, and let

(1) $$\delta = 1 - u_S(a) - u_S(b) > 0.$$

For convenience we divide the proof of Case 1 into three steps.

Step A: *Construction of* ϕ. We shall construct a ϕ_δ—call it ϕ—which approximately maps S onto an annulus.

Let u_i be the continuous function on S which is harmonic on $\operatorname{int} S$ and satisfies:

$$u_i|\partial D_i \equiv 1, \qquad u_i|\partial S \setminus \partial D_i \equiv 0; \qquad i = 1, 2, \cdots, n.$$

Note that $u_S = \sum_{i=1}^n u_i$ by Lemma 3. Each u_i extends to be harmonic in a neighborhood of S by the reflection principle. In this way we obtain an open set $\mathcal{O} \supseteq S$ which we may take small enough that

(i) $\sum_{i=1}^n |u_i(z)| < 2$ for $z \in \mathcal{O}$;

(ii) $u_S(z) > 1$ for $z \in \mathcal{O} \cap \operatorname{int} D_i$, $i = 1, 2, \cdots, n$;

(iii) $u_i(z) < 0$ for $z \in \mathcal{O} \setminus \Delta$, $i = 1, 2, \cdots, n$.

We may also assume that $\partial\mathcal{O}$ consists of circles, one in each component of $\mathbf{C} \setminus S$.

Let $\gamma_0\colon [0, 1] \to \mathcal{O}$ be a positively oriented circle contained in $\mathcal{O} \setminus \Delta$. Similarly, let γ_j be a positively oriented circle contained in $\mathcal{O} \cap \operatorname{int} D_j$, $j = 1, 2, \cdots, n$. To avoid confusion we will distinguish between the map γ_j and its image which we will denote by Γ_j, $j = 0, 1, \cdots, n$. If γ is any closed curve in $\mathcal{O}$, then γ is homologous in $\mathcal{O}$ to $\sum_{j=1}^n c_j \gamma_j$ for some integers $c_1, \cdots, c_n$.

Let v_S be the (multiple-valued) conjugate harmonic function to u_S, and let v_i be the harmonic conjugate of u_i, $i = 1, 2, \cdots, n$. Define

$$\alpha_j = \frac{1}{2\pi} \operatorname*{var}_{\gamma_j} v_S, \qquad \omega_{ij} = \frac{1}{2\pi} \operatorname*{var}_{\gamma_j} v_i.$$

Observe that $\det((\omega_{ij})) \neq 0$. If not, there would be real constants c_i, not all zero, such that

$$\sum_{i=1}^n c_i \omega_{ij} = 0 \quad \text{for } j = 1, 2, \cdots, n.$$

Then $f = \sum_{i=1}^n c_i(u_i + iv_i)$ would be a nonconstant single-valued analytic function on $\mathcal{O}$, with

$$f(\partial S) \subseteq \{\operatorname{Re} z = 0\} \cup \{\operatorname{Re} z = c_1\} \cup \cdots \cup \{\operatorname{Re} z = c_n\}.$$

As this is absurd, we conclude that there are indeed constants η_{ki}, $1 \leq k, i \leq n$, such that $\sum_{i=1}^n \eta_{ki}\omega_{ij} = \delta_k^j$, $1 \leq j, k \leq n$.

Now choose real numbers $\beta_1, \cdots, \beta_n$ in such a way that $\alpha_j + \beta_j$ is rational for each j, making sure that the β_j are sufficiently small that if we let $d_i = \Sigma_{k=1}^n \beta_k \eta_{ki}$, then

$$K = 2 \max_i |d_i| < \delta/2. \tag{2}$$

Now we define u on $\mathcal{O}$ by

$$u = u_S + \sum_{i=1}^n d_i u_i = \sum_{i=1}^n (1 + d_i) u_i. \tag{3}$$

The harmonic conjugate of u is $v = v_S + \Sigma_{i=1}^n d_i v_i$, so

$$\frac{1}{2\pi} \operatorname*{var}_{\gamma_j} v = \frac{1}{2\pi} \operatorname*{var}_{\gamma_j} v_S + \frac{1}{2\pi} \sum_{i=1}^n d_i \operatorname*{var}_{\gamma_j} v_i = \alpha_j + \sum_{i=1}^n d_i \omega_{ij}$$

$$= \alpha_j + \sum_{i=1}^n \sum_{k=1}^n \beta_k \eta_{ki} \omega_{ij} = \alpha_j + \sum_{k=1}^n \beta_k \delta_k^j = \alpha_j + \beta_j.$$

Choose a positive integer N large enough that $N(\alpha_j + \beta_j)$ is an integer for $j = 1, 2, \cdots, n$, and let $\phi = e^{-N(u+iv)}$. By construction, ϕ is a single-valued analytic function on $\mathcal{O}$.

Step B: *Properties of* ϕ. Let $B = \{\zeta \in \mathbf{C} \mid 1 \geq |\zeta| \geq e^{-N(1-K)}\}$. We claim that $\phi^{-1}(B) \subseteq S$. Note that for $z \in \phi^{-1}(B)$ we have $0 \leq u(z) \leq 1 - K$, and suppose that $z \in \mathcal{O} \setminus S$. There are two cases to consider.

If $z \in \mathcal{O} \setminus \Delta$, then $u_i(z) < 0$ for $i = 1, 2, \cdots, n$ by condition (iii) above. Note that $\delta \leq 1$ by equation (1), so that

$$u(z) = \sum_{i=1}^n (1 + d_i) u_i(z) < \sum_{i=1}^n (1 - \delta/4) u_i(z) < 0$$

by equations (2) and (3). Thus $z \notin \phi^{-1}(B)$.

If $z \in \operatorname{int} D_j \cap \mathcal{O}$, then

$$u(z) = u_S(z) + \sum_{i=1}^n d_i u_i(z)$$

$$\geq u_S(z) - \sum_{i=1}^n |d_i||u_i(z)| > u_S(z) - K/2 \cdot 2 \geq 1 - K;$$

this follows from equations (2) and (3) and conditions (i) and (ii) above. Again, $z \notin \phi^{-1}(B)$.

In particular, we have established that $\phi^{-1}(\{|\zeta| = 1\})$ is contained in S. But if $z \in S\setminus\partial\Delta$, then $u_i(z) \geq 0$ for $i = 1, 2, \cdots, n$, and $u_S(z) > 0$ as noted earlier, so that

$$u(z) = \sum_{i=1}^{n} (1 + d_i)u_i(z) \geq (1 - \delta/4)u_S(z) > 0.$$

Thus $\phi^{-1}(\partial\Delta) \subseteq \partial\Delta$.

Now let U be the subset of $\mathcal{O}$ bounded by the images Γ_j of the maps γ_j, $j = 0, 1, \cdots, n$. Let

$$R = \inf_{z \in \Gamma_0} e^{-Nu(z)}, \qquad r = \sup_{z \in \Gamma_j;\ 1 \leq j \leq n} e^{-Nu(z)}.$$

$\Gamma_0 \subseteq \mathcal{O}\setminus\Delta$, so by the above $R > 1$. Similarly, $\Gamma_j \subseteq \mathcal{O} \cap \operatorname{int} D_j$, so that $r < e^{-N(1-K)} < 1$. Let

$$U' = \{z \in U|\ R > e^{-Nu(z)} > r\},$$

and note that $\phi(U') \supseteq B$, since $\phi(\partial U') = \{|\zeta| = r\} \cup \{|\zeta| = R\}$.

For $\zeta \in \phi(\mathcal{O})$ let $m(\zeta)$ denote the number of times including multiplicities that ϕ takes on the value ζ in U. $m(\zeta)$ is constant on each component of $\phi(\mathcal{O})\setminus \bigcup_{j=0}^{n} \phi(\Gamma_j)$. By choice of r and R, the set $\{\zeta \in \mathbf{C}|\ r < |\zeta| < R\}$ is contained in a component of $\phi(\mathcal{O})\setminus \bigcup_{j=0}^{n} \phi(\Gamma_j)$. Thus $m(\zeta)$ is constant on $\phi(U')$; we use m to denote this constant value. $m > 0$ since if $z_0 \in \partial\Delta$, then $r < 1 = |\phi(z_0)| < R$.

We conclude that for all $\zeta \in \phi(U')$, ζ has precisely m inverses in U' under ϕ. Call these inverses $\psi_1(\zeta), \cdots, \psi_m(\zeta)$ in any order.

Step C: *Reduction to Lemma* 4. Recall that we are trying to show that a given polynomial P which vanishes at (a, b) also has a zero on X_S. We are now ready to use P to define a function $Q \in R(X_B)$. In fact, let

$$Q(\zeta, \eta) = \prod_{i,j=1}^{m} P(\psi_i(\zeta), \psi_j(\eta)), \qquad (\zeta, \eta) \in \phi(U') \times \phi(U').$$

Note that Q is well defined since it is independent of the numbering of the ψ_i. Fix η. Except for ζ in a discrete subset of $\phi(U')$, ζ will have m distinct inverse images in U' under ϕ. For such ζ one may renumber the ψ_i near ζ to make them analytic near ζ. Thus $Q(\zeta, \eta)$ is analytic in ζ except possibly on a discrete set. For any of the isolated points ζ_0 with less than m distinct inverse images, $\zeta \to \prod_{i,j=1}^{m} P(\psi_i(\zeta), \psi_j(\eta))$ is analytic and bounded in a deleted neighborhood of ζ_0, and so extends to be analytic at ζ_0. Similarly, for fixed ζ the function $\eta \to \Pi_{i,j=1}^m P(\psi_i(\zeta), \psi_j(\eta))$ is analytic on $\phi(U')$, so that Q is analytic on $\phi(U') \times \phi(U')$. Now it was noted previously that $B \subseteq \phi(U')$, so $X_B \subseteq B \times B \subseteq \phi(U') \times \phi(U')$. Since $B \times B$ is rationally convex, $Q \in R(X_B)$.

Observe now that

$$\begin{aligned}
|\phi(a)\phi(b)| &= |\exp\{-N(u(a) + iv(a)) - N(u(b) + iv(b))\}|.\\
&= \exp\{-N(u(a) + u(b))\}\\
&\geq \exp\left\{-N\left[\sum_{i=1}^{n} (1 + |d_i|)(u_i(a) + u_i(b))\right]\right\} && \text{by (3)}\\
&\geq \exp\{-N[(1 + K/2)(u_S(a) + u_S(b))]\} && \text{by (2)}\\
&\geq \exp\{-N[1 - \delta + K/2]\} && \text{by (1)}\\
&\geq \exp\{-N[1 - K]\} && \text{by (2).}
\end{aligned}$$

By Lemma 4, $h_r(X_B) = \{(\zeta, \eta) \in \Delta^2 |\ 1 \geq |\zeta\eta| \geq e^{-N(1-K)}\}$, so we have that $(\phi(a), \phi(b)) \in h_r(X_B)$. By definition of the ψ_i there exist i_0 and j_0 such that $\psi_{i_0}(\phi(a)) = a$ and $\psi_{j_0}(\phi(b)) = b$. (Note that $\psi_i(\phi(a))$ and $\psi_j(\phi(b))$ are well defined for all i and j, since $(\phi(a), \phi(b)) \in h_r(X_B)$ implies $\phi(a), \phi(b) \in B$, and so by the fact that $B \subseteq \phi(U')$ we have $\phi(a), \phi(b) \in \phi(U')$ as desired.) Thus,

$$Q(\phi(a), \phi(b)) = P(a, b) \prod_{i \neq i_0 \text{ or } j \neq j_0} P(\psi_i(\phi(a)), \psi_j(\phi(b))) = 0.$$

It follows that there is some point $(\zeta_0, \eta_0) \in X_B$ such that $Q(\zeta_0, \eta_0) = 0$, as was shown in the proof of Lemma 5. Hence

$$\prod_{i,j=1}^{m} P(\psi_i(\zeta_0), \psi_j(\eta_0)) = 0,$$

so that for some i_1, j_1 we have $P(\psi_{i_1}(\zeta_0), \psi_{j_1}(\eta_0)) = 0$. As noted earlier, $\phi^{-1}(\partial\Delta) \subseteq \partial\Delta$ and $\phi^{-1}(B) \subseteq S$, so $(z_0, w_0) = (\psi_{i_1}(\zeta_0), \psi_{j_1}(\eta_0)) \in X_S$. Thus P has the zero $(z_0, w_0) \in X_S$ as claimed.

Case 2. Suppose that $u_S(a) + u_S(b) > 1$. Let

$$\delta = u_S(a) + u_S(b) - 1 > 0. \tag{1'}$$

Proceed to define ϕ exactly as in Case 1. Now, however, let

$$B' = \{\zeta \in \mathbf{C}|\ 1 \geq |\zeta| \geq e^{-N(1+K)}\},$$

a larger annulus than before. In fact, if $z \in S$ then $\phi(z) \in B'$ because

$$|\phi(z)| = \exp\{-Nu(z)\} = \exp\left\{-N\left(u_S(z) + \sum_{i=1}^{n} d_i u_i(z)\right)\right\}$$

$$\geq \exp\{-N(u_S(z) + K/2)\} \geq \exp\{-N(1 + K/2)\}$$

by using equations (2) and (3) which are still valid.

Now define $\Phi(z, w) = (\phi(z), \phi(w))$ for $(z, w) \in \mathcal{O} \times \mathcal{O}$. Φ is analytic on $\mathcal{O} \times \mathcal{O}$, so $z \circ \Phi$ and $w \circ \Phi$ are in $R(X_S)$. We have

$$\begin{aligned} |\phi(a)\phi(b)| &= \exp\{-N(u(a) + u(b))\} \\ &\leq \exp\left\{-N\left(u_S(a) + u_S(b) - \sum_{i=1}^{n} |d_i|(u_i(a) + u_i(b))\right)\right\} && \text{by (3)} \\ &\leq \exp\{-N(1 + \delta - K)\} && \text{by (1'), (2)} \\ &< \exp\{-N(1 + K)\} && \text{by (2).} \end{aligned}$$

Thus by Lemma 4, $\Phi(a, b) \notin h_r(X_{B'})$. The remark at the end of the preceding paragraph shows that $\phi(S) \subseteq B'$; as in Case 1, $\phi(\partial\Delta) = \partial\Delta$, so $\Phi(X_S) = X_{\phi(S)} \subseteq X_{B'}$. Thus $\Phi(a, b) \notin h_r(\Phi(X_S))$. By Lemma 5, $(a, b) \notin h_r(X_S)$.

Case 3. Suppose, finally, that $u_S(a) + u_S(b) = 1$. We may assume that $u_S(a) > 0$. Choose $a_n \in S$ with $u_S(a_n) < u_S(a)$ and $a_n \to a$. We have $(a_n, b) \to (a, b)$, and each $(a_n, b) \in h_r(X_S)$, since $u_S(a_n) + u_S(b) < 1$. Since $h_r(X_S)$ is closed, $(a, b) \in h_r(X_S)$.

Given an arbitrary set S, one can find a sequence of regions S_n with properties (i), (ii) and (iii) in the definition of u_S following Lemma 1. For each n, int S_n is conformally equivalent to a domain bounded by circles; the equivalence extends to a continuous 1 to 1 map on S_n. By the corollary to Lemma 5 and the special case above, the theorem holds for each S_n.

Since $S \subseteq S_n$ for each n, $h_r(X_S) \subseteq \bigcap_{n=1}^{\infty} h_r(X_{S_n})$. Conversely, suppose that $p \in h_r(X_{S_n})$ for all n. If P is any polynomial which vanishes at p, then there is some point $p_n \in X_{S_n}$ such that $P(p_n) = 0$. By compactness there is a subsequence p_{n_k} converging to $p_0 \in \bigcap_{n=1}^{\infty} X_{S_n} = X_S$, and clearly $P(p_0) = 0$. Thus $p \in h_r(X_S)$. So we have $h_r(X_S) = \bigcap_{n=1}^{\infty} h_r(X_{S_n})$. Now $u_S(z) = \lim_{n\to\infty} u_{S_n}(z)$ for $z \in S$, and the u_{S_n} are monotone increasing, so

$$h_r(X_S) = \bigcap_{n=1}^{\infty} h_r(X_{S_n}) = \bigcap_{n=1}^{\infty} \{(z, w) \in S_n \times S_n \mid u_{S_n}(z) + u_{S_n}(w) \leq 1\}$$

$$= \{(z, w) \in S \times S \mid u_S(z) + u_S(w) \leq 1\}$$

and the theorem is proven.

Note 1. The result can, of course, be generalized somewhat. For example, by modifying Lemma 4 and the proof of Theorem 1, one can show

$$\text{given } S_1, S_2, \cdots, S_n \text{ then } h_r(S_1 \times S_2 \times \cdots \times S_n \cap \partial\Delta^n)$$
$$= \{z \in S_1 \times S_2 \times \cdots \times S_n \mid u_{S_1}(z_1) + \cdots + u_{S_n}(z_n) \leq n - 1\}.$$

Note 2. The restriction to sets S which are connected is not serious, since more general subsets of Δ are of no additional interest. In fact, if $\partial\Delta \subseteq T \subseteq \Delta$ and T is closed, let S be the component of $\partial\Delta$ in T. One readily verifies that

$$h_r(X_T) = h_r(X_S) \cup X_T, \tag{4}$$

so that the other components of T contribute nothing to $h_r(X_T)\setminus X_T$. We outline a proof.

Choose nonempty compact sets $K_n \subseteq \Delta\setminus S$ such that $K_{n+1} \supseteq K_n$ and $\bigcup_{n=1}^{\infty} K_n = \Delta\setminus S$. One can then choose smoothly bounded compact sets C_n, D_n such that $T \subseteq C_n \cup D_n$, $S \subseteq C_n$, $K_n \subseteq D_n$, and $C_n \cap D_n = \emptyset$. Let l_n be a finite union of line segments which meet C_n and D_n precisely at their endpoints, chosen so that the compact set S_n defined by $S_n = C_n \cup D_n \cup l_n$ is connected.

Observe that $u_{S_n} \equiv 1$ on $D_n \cup l_n$. As remarked earlier, $u_{S_n} > 0$ on $S_n\setminus\partial\Delta$, so, by Theorem 1, $h_r(X_{S_n}) = h_r(X_{C_n}) \cup X_{S_n}$. The result (4) follows by using the facts that $T \subseteq S_n$ and that $\bigcap_{n=1}^{\infty} C_n = S$.

Example. In [7], Wermer shows that $h_r(X_S)$ does not "contain analytic structure" precisely when $\operatorname{int} S = \emptyset$. He proceeds to construct a specific S such that $h_r(X_S) \neq X_S$, although $\operatorname{int} S = \emptyset$. Using Theorem 1 we have an alternative method for constructing such an S.

If $|a| < 1$ and $0 < r < 1 - |a|$, let $D(a, r) = \{|z - a| < r\}$. Then $u_{\Delta\setminus D(a,r)}(z) \to 0$ as $r \to 0$ if $z \in \Delta\setminus\{a\}$. (For example, if $a = 0$, then $u_{\Delta\setminus D(a,r)}(z) = \log|z|/\log r$ and this assertion is clear; the general case can be essentially reduced to the case $a = 0$ by appropriate linear fractional transformations.)

Let $\{a_j\}$ be a countable dense subset of $\operatorname{int}\Delta$, and let b be any other point of $\operatorname{int}\Delta$. Take D_1 to be an open disk centered at a_1, such that $\overline{D}_1 \subseteq \operatorname{int}\Delta$ and $u_{\Delta\setminus D_1}(b) \leq 1/4$; this is possible by the preceding remark. Suppose that

$D_1, D_2, \cdots, D_{n-1}$ have been chosen. Let a be the first of the a_j which is not in $\bigcup_{i=1}^{n-1} \overline{D}_i$. Choose an open disk D_n with center a such that $\overline{D}_n \subseteq \operatorname{int} \Delta \backslash \bigcup_{i=1}^{n-1} \overline{D}_i$ and $u_{\Delta \backslash D_n}(b) \leq 1/2^{n+1}$. Let $S = \Delta \backslash \bigcup_{i=1}^{\infty} D_i$. By construction $\operatorname{int} S = \varnothing$, and $u_S(b) \leq \sum_{i=1}^{\infty} u_{\Delta \backslash D_i}(b) \leq 1/2$, so that $(b, b) \in h_r(X_S) \backslash X_S$ by Theorem 1.

Note. Theorem 1 also shows that one must be careful in the choice of S to guarantee $h_r(X_S) \neq X_S$. Suppose, for example, that S is a connected compact subset of Δ containing $\partial\Delta$, such that the only Jensen measures for $R(S)$ are point masses. Then X_S is rationally convex, even when $R(S) \neq C(S)$. To see this, let $\{S_n\}_{n=1}^{\infty}$ be as in the definition of u_S.

Given $z \in S \backslash \partial\Delta$, let μ_n be harmonic measure for z as a point of S_n; each μ_n is then a Jensen measure for $R(S_n)$. Each μ_n may be regarded as a measure on Δ, and so, by passing to a subsequence, we may assume that $\{\mu_n\}$ converges weak-star on Δ to a measure μ with support contained in S. μ represents z for $R(S)$ and is in fact a Jensen measure as we will now show.

Let $f \in R(S)$. Choose f_m analytic on a neighborhood of S so that f_m converges to f uniformly on S as $m \to \infty$. If n is large, $f_m \in R(S_n)$ and so

$$\log |f_m(z)| \leq \int \log |f_m| \, d\mu_n.$$

Thus for any $\delta > 0$ we have

$$\log |f_m(z)| \leq \int \log (|f_m| + \delta) \, d\mu_n$$

and so

$$\log |f_m(z)| \leq \lim_{\delta \downarrow 0} \lim_{n \to \infty} \int \log (|f_m| + \delta) \, d\mu_n = \int \log |f_m| \, d\mu.$$

Let $\epsilon > 0$; then $\log(|f_m| + \epsilon)$ converges uniformly on S to $\log(|f| + \epsilon)$ as $m \to \infty$, and so

$$\lim_{m \to \infty} \int \log (|f_m| + \epsilon) \, d\mu = \int \log (|f| + \epsilon) \, d\mu.$$

On the other hand we have

$$\lim_{m \to \infty} \int \log (|f_m| + \epsilon) \, d\mu \geq \varlimsup_{m \to \infty} \int \log |f_m| \, d\mu$$

$$\geq \lim_{m \to \infty} \log |f_m(z)| = \log |f(z)|.$$

Combining these results and applying monotone convergence we see that

$$\int \log|f|\,d\mu = \lim_{\epsilon \downarrow 0} \int \log(|f| + \epsilon)\,d\mu \geq \log|f(z)|.$$

Thus μ is a Jensen measure for the point z with respect to $R(S)$. By our assumption on S, μ is the unit point mass at z.

Choose $\chi \in C(\Delta)$ such that $0 \leq \chi \leq 1$, $\chi(z) = 1$, and $\operatorname{supp}\chi \subseteq \operatorname{int}\Delta$. Then

$$1 \geq u_S(z) = \lim_{n\to\infty} u_{S_n}(z) = \lim_{n\to\infty} \int_{\partial S_n \setminus \partial\Delta} 1\,d\mu_n$$
$$\geq \lim_{n\to\infty} \int_{\partial S_n} \chi\,d\mu_n = \int_S \chi\,d\mu = \chi(z) = 1.$$

Suppose now that $(z, w) \in h_r(X_S)$ and $z \in S\setminus\partial\Delta$. By Theorem 1 we have $u_S(w) \leq 1 - u_S(z) = 0$. As noted earlier, $u_S(w) = 0$ implies $w \in \partial\Delta$. Thus $h_r(X_S) = X_S$.

Note. It follows rather readily from Theorem 1 that if $h_r(X_S) \neq X_S$, then the 4-dimensional measure of $h_r(X_S)\setminus X_S$ is positive (in contrast to that of X_S which, consisting of two copies of $S \times \partial\Delta$, has only positive 3-dimensional measure). To see this, extend u_S to all of Δ by

$$u_S(z) = 1 \quad \text{if } z \in \Delta\setminus S.$$

Then u_S is superharmonic on $\operatorname{int}\Delta$ (see Lemma 6 below), so that the function g defined on $\operatorname{int}\Delta \times \operatorname{int}\Delta$ by $g(z, w) = u_S(z) + u_S(w)$ is superharmonic. If $(z_0, w_0) \in h_r(X_S)\setminus X_S$, then $(z_0, w_0) \in \operatorname{int}\Delta \times \operatorname{int}\Delta$ and, by Theorem 1, $u_S(z_0) + u_S(w_0) \leq 1$. Thus if B_ϵ is any small ball around (z_0, w_0) and if $\overline{m}$ denotes 4-dimensional Lebesgue measure on $\mathbf{C}^2$ we have by the superharmonicity of g:

$$\frac{1}{\text{volume } B_\epsilon} \int_{B_\epsilon} g\,d\overline{m} \leq 1.$$

This implies that $\{(z, w) \in B_\epsilon \mid g(z, w) = u_S(z) + u_S(w) \leq 1\}$, which by Theorem 1 is contained in $h_r(X_S)$, has positive 4-dimensional measure.

A more careful analysis leads to a significant sharpening of this result; in fact, we are able to show that some of the "Gleason parts" for $R(X_S)$ have positive 4-dimensional measure. The concept of "Gleason part" has been useful in the study of analytic structure in the maximal ideal space of a uniform algebra (see, e.g., [3, Chapter VI]). If T is a compact set in $\mathbf{C}^n$, the Gleason part (for the algebra $R(T)$ "=" $R(h_r(T))$) of a point $p \in h_r(T)$ is the equivalence class of p under the equivalence relation:

$$p \sim q \text{ if } \sup\left\{|f(p) - f(q)| \,\middle|\, f \in R(T) \text{ and } \sup_T |f| \leq 1\right\} < 2.$$

(Thus, for instance, if $h_r(T)$ contains a connected analytic variety V, a normal family argument shows that the points of V belong to a single part of $h_r(T)$.)

Theorem 2. *If $h_r(X_S)\setminus X_S \neq \emptyset$, then there is a point $(z_0, w_0) \in h_r(X_S)\setminus X_S$ with $u_S(z_0) + u_S(w_0) < 1$. Any such point belongs to a Gleason part of $h_r(X_S)$ with positive 4-dimensional measure.*

The proof of Theorem 2 depends on several lemmas, for which we establish the following notation:

m = 2-dimensional Lebesgue measure on the plane $\mathbf{C}$;

$S^r = \{z \in S \mid u_S(z) \leq r\}$, $0 < r \leq 1$ (note that S^r is closed since by Lemma 6 below, u_S is superharmonic).

Lemma 6. *u_S is superharmonic on* $\operatorname{int}\Delta$ *(as above, we consider u_S extended to Δ by $u_S \equiv 1$ on $\Delta\setminus S$).*

Proof. Take $\{S_n\}_{n=1}^{\infty}$ as in the definition of u_S. One verifies that each u_{S_n} is superharmonic on $\operatorname{int}\Delta$ by considering three cases:

(i) if $z \in \Delta\setminus S_n$, then $u_{S_n} \equiv 1$ near z;

(ii) if $z \in \operatorname{int} S_n$, then u_{S_n} is harmonic near z by definition;

(iii) if $z \in \partial S_n \cap \operatorname{int}\Delta$, and $r < 1 - |z|$, then

$$u_{S_n}(z) = 1 = \frac{1}{2\pi}\int_0^{2\pi} d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} u_{S_n}(z + re^{i\theta})\, d\theta.$$

Thus u_S is the pointwise limit on $\operatorname{int}\Delta$ of the monotone increasing sequence of superharmonic functions $\{u_{S_n}\}$. So u_S is superharmonic on $\operatorname{int}\Delta$.

Corollary. *If $0 < u_S(z) < 1$, there is a point z' near z where $u_S(z') < u_S(z)$.*

Proof. If $z \in \operatorname{int} S$, the statement is trivial since u_S is harmonic near z. So suppose that $z \in \partial S\setminus\partial\Delta$. Then we can find r arbitrarily small so that the set $E = \{\theta \mid z + re^{i\theta} \in \Delta\setminus S\}$ has positive 1-dimensional measure. If we also had $u(z + re^{i\theta}) \geq u(z)$ for all θ, we would have

$$\frac{1}{2\pi}\int_{[0,2\pi]} u(z + re^{i\theta})\, d\theta \geq \frac{1}{2\pi}\int_E 1\, d\theta + \frac{1}{2\pi}\int_{[0,2\pi]\setminus E} u(z)\, d\theta > u(z).$$

This contradiction establishes that $u(z + re^{i\theta}) < u(z)$ for some θ.

This corollary verifies the easy half of Theorem 2, since $(z, w) \in h_r(X_S)\setminus X_S$ and $u_S(z) + u_S(w) = 1$ imply $0 < u_S(z) < 1$. Applying the corollary to z, we have z' near z with $u_S(z') < u_S(z)$. Then (z', w) has the property that $u_S(z') + u_S(w) < 1$, and, of course, by Theorem 1 $(z', w) \in h_r(X_S)\setminus X_S$.

Lemma 7. *Fix r, $0 < r \leq 1$. If $0 < u_S(z_0) < r$, then z_0 is not a peak point for $R(S^r)$.*

Proof. Again take $\{S_n\}$ as in the definition of u_S. Since $u_{S_n}(z_0) \leq u_S(z_0)$ for all n, $z_0 \in (S_n)^r$ for all n. Let μ_n be a probability measure supported on ∂S_n^r such that for all functions f harmonic in a neighborhood of S_n^r we have

$$\int_{\partial S_n^r} f\, d\mu_n = f(z_0).$$

By passing to a subsequence we may assume that μ_n converges weak-star to a probability measure μ on ∂S^r which then has the property that, for all f harmonic in a neighborhood of S^r, $\int_{\partial S^r} f\, d\mu = f(z_0)$. It follows that μ represents z_0 for $R(S^r)$.

Choose $\chi \in C(\Delta)$ such that $0 \leq \chi \leq 1$, $\chi(z_0) = 1$, and $\operatorname{supp} \chi \subseteq \operatorname{int} \Delta$. Then

$$\int_{\partial S^r} \chi\, d\mu = \lim_{n \to \infty} \int_{\partial S_n^r} \chi\, d\mu_n \leq \lim_{n \to \infty} \int_{\partial S_n^r} \frac{u_{S_n}}{r}\, d\mu_n$$

$$= \lim_{n \to \infty} \frac{u_{S_n}(z_0)}{r} = \frac{u_S(z_0)}{r} < 1,$$

where we have made use of the fact that $u_{S_n} = r$ on $\partial S_n^r \backslash \partial\Delta$ (if $u_{S_n}(z) < r$, then $u_{S_n} < r$ near z). So μ is not a point mass and therefore z_0 is not a peak point for $R(S^r)$.

Lemma 8. *Let $(z_0, w_0) \in h_r(X_S) \backslash (X_S)$, and suppose that $z_0 \sim z_1$ in $S^{1-u_S(w_0)}$. Then $(z_1, w_0) \in h_r(X_S) \backslash X_S$, and $(z_0, w_0) \sim (z_1, w_0)$ in $h_r(X_S)$. Similarly, if $w_1 \sim w_0$ in $S^{1-u_S(z_1)}$, then $(z_1, w_1) \in h_r(X_S) \backslash X_S$ and $(z_1, w_1) \sim (z_1, w_0)$ in $h_r(X_S)$.*

Proof. Let

$$K = h_r(X_S) \cap \{(z, w) | \ w = w_0\}$$

$$= \{(z, w_0) \in \Delta^2 | \ u_S(z) \leq 1 - u_S(w_0)\} = S^{1-u_S(w_0)} \times \{w_0\}.$$

K is a compact subset of the plane $\{w = w_0\}$, and is nonempty since $(z_0, w_0) \in K$.

Since we have assumed that $z_1 \in S^{1-u_S(w_0)}$, we have $u_S(z_1) \leq 1 - u_S(w_0)$, so $(z_1, w_0) \in h_r(X_S)$, whence $(z_1, w_0) \in K$.

Let $F_n \in R(h_r(X_S))$, $|F_n| \leq 1$, and $F_n(z_0, w_0) \to +1$. Then $F_n(z, w_0) \in R(K)$, $|F_n(z, w_0)| \leq 1$, so by the assumption that $z_0 \sim z_1$ in $S^{1-u_S(w_0)}$, $F_n(z_1, w_0) \to +1$.

Thus $(z_0, w_0) \sim (z_1, w_0)$ in $h_r(X_S)$. The second half of the lemma follows by applying the same argument with the roles of the coordinates reversed.

Proof of Theorem 2. Choose δ so that $0 < \delta < 1 - u_S(z_0) - u_S(w_0)$. Let

$$T_1 = \{z \in S^{1-u_S(w_0)} \mid u_S(z) \le u_S(z_0) + \delta,\ z \sim z_0 \text{ in } S^{1-u_S(w_0)}\},$$

$$T_2 = \{w \in S^{1-u_S(z_0)-\delta} \mid w \sim w_0 \text{ in } S^{1-u_S(z_0)-\delta}\}.$$

If $z \in T_1$, $(z, w_0) \sim (z_0, w_0)$ in $h_r(X_S)$ by Lemma 8. We also have $u_S(z) \le u_S(z_0) + \delta$, so that $S^{1-u_S(z)} \supseteq S^{1-u_S(z_0)-\delta}$. Hence if $w \in T_2$, then $w \sim w_0$ in $S^{1-u_S(z)}$. By Lemma 8, $(z, w) \sim (z, w_0)$ in $h_r(X_S)$. Thus $(z, w) \sim (z_0, w_0)$ in $h_r(X_S)$, and so the part of $R(h_r(X_S))$ to which (z_0, w_0) belongs contains $T_1 \times T_2$. Since $R(X_S)$ can be identified with $R(h_r(X_S))$, the theorem will be proved if we show that $m(T_1) > 0$ and $m(T_2) > 0$.

For $0 < R < 1 - |z_0|$, let

$$E_R = \{z \in S \mid u_S(z) > u_S(z_0) + \delta\} \cap \{|z - z_0| < R\}.$$

By the superharmonicity of u_S, we have

$$u_S(z_0) \ge \frac{1}{\pi R^2} \int_{|z-z_0|<R} u_S(z)\, dm \ge \frac{1}{\pi R^2} \int_{E_R} (u_S(z_0) + \delta)\, dm$$

$$= [u_S(z_0) + \delta]\, m(E_R)/\pi R^2.$$

Thus $m(E_R) \le \pi R^2 (u_S(z_0)/(u_S(z_0) + \delta))$. Note also that $u_S(z_0) < 1 - u_S(w_0)$, so by Lemma 7, z_0 is not a peak point of $R(S^{1-u_S(w_0)})$.

Let K be a compact subset of $\mathbf{C}$, and suppose that $x \in K$ is not a peak point for $R(K)$. Given $\epsilon > 0$, let

$$P_\epsilon = \left\{ y \in K \,\middle|\, \sup \left\{ |f(x) - f(y)| \,\middle|\, f \in R(K),\ \sup_K |f| \le 1 \right\} < \epsilon \right\}.$$

A theorem due to Browder says that

$$\lim_{n\to\infty} \frac{m(P_\epsilon \cap \{y \in \mathbf{C} \mid |x - y| < 1/n\})}{m(\{y \in \mathbf{C} \mid |x - y| < 1/n\})} = 1$$

(see [13], or [9, pp. 175–177]). In particular, if $\epsilon = 1$ this says that the part of $R(K)$ to which x belongs has density one at x. (This consequence of Browder's theorem was also proved by Melnikov [14] and by Wilken.) Applying this to $z_0 \in S^{1-u_S(w_0)}$, we obtain

$$m(\{z \in S^{1-u_S(w_0)} \mid |z - z_0| < R,\ z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) > \pi R^2(u_S(z_0)/(u_S(z_0) + \delta))$$

for R sufficiently small.

Combining this result with the above we see that for R small and positive:

$$\begin{aligned} &m(T_1 \cap \{|z - z_0| < R\}) \\ &\quad = m\,(\{z \mid |z - z_0| < R,\ u_S(z) \le u_S'(z_0) + \delta,\ z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) \\ &\quad = m(\{|z - z_0| < R\}) - m(E_R) \\ &\qquad - m(\{z \mid |z - z_0| < R,\ u_S(z) \le u_S(z_0) + \delta, \text{ but not } z \sim z_0 \text{ in } S^{1-u_S(w_0)}\}) \\ &\quad > \pi R^2 - \pi R^2\left(\frac{u_S(z_0)}{u_S(z_0) + \delta}\right) - \left[\pi R^2 - \pi R^2\left(\frac{u_S(z_0)}{u_S(\dot{z}_0) + \delta}\right)\right] = 0. \end{aligned}$$

In particular, $m(T_1) > 0$.

Finally, $u_S(w_0) < 1 - u_S(z_0) - \delta$ by definition of δ, so by Lemma 7 w_0 is not a peak point for $R(S^{1-u_S(z_0)-\delta})$. Thus $m(T_2) > 0$ since nontrivial parts have positive measure. This and the last corollary complete the proof of Theorem 2.

2. Let B denote the closed unit ball in $\mathbf{C}^2 = \{(z, w) \mid |z|^2 + |w|^2 \le 1\}$. If T is an arbitrary compact subset of the open unit disk, we associate with T a subset of the 3-sphere $\{|z|^2 + |w|^2 = 1\}$ by defining

$$\widetilde{X}_T = \{(z, w) \in \partial B \mid z \in T\}.$$

Two observations about $R(\widetilde{X}_T)$ can be made immediately. First, since it is well known that if X is a compact subset of $\mathbf{C}^n$ then $R(X)$ is generated by $n + 1$ functions (for a proof see Lemma 11), the algebra $R(\widetilde{X}_T)$ is finitely generated. Secondly, each point of $\widetilde{X}_T$ is a peak point for $R(\widetilde{X}_T)$; this is in itself a trivial observation, since if Y is any compact subset of ∂B then each point $(z_0, w_0) \in Y$ is a peak point for $P(Y)$—hence for $R(Y)$—because $P(z, w) = (\bar{z}_0 z + \bar{w}_0 w + 1)/2$ peaks at (z_0, w_0). (This explicit peaking function was pointed out to me by several people.) This latter fact, however, becomes particularly significant in conjunction with the following result:

Theorem 3. *Suppose that the only Jensen measures for $R(T)$ are trivial, and $R(T) \neq C(T)$. Then*

(i) *the maximal ideal space of $R(\widetilde{X}_T)$ is $\widetilde{X}_T$, i.e., $\widetilde{X}_T$ is rationally convex;*

(ii) $R(\widetilde{X}_T) \neq C(\widetilde{X}_T)$.

Thus for such a T properties I, II, III mentioned in the introduction are satisfied by the algebra $A = R(\tilde{X}_T)$. The proof of Theorem 3 can be divided into two lemmas.

Lemma 9. *Suppose that T is a compact subset of the open unit disk, and the only Jensen measures for $R(T)$ are trivial. Then $\tilde{X}_T$ is rationally convex.*

Proof. Suppose $(\alpha, \beta) \in h_r(\tilde{X}_T)$. Let μ be a Jensen measure on $\tilde{X}_T$ for the homomorphism corresponding to evaluation at (α, β). We assert that

$$\operatorname{supp} \mu \subseteq \{(z, w) \in \tilde{X}_T \mid z = \alpha\}.$$

The latter set is a circle which we denote by Γ_α.

Observe that $\alpha \in T$, since otherwise $z - \alpha$ vanishes at (α, β) but has no zero on $\tilde{X}_T$. Define a measure $\bar{\mu}$ on T by

$$\int_T g \, d\bar{\mu} = \int_{\tilde{X}_T} g(z) \, d\mu(z, w), \qquad g \in C(T).$$

Let $f \in R(T)$; then since μ is a Jensen measure for (α, β)

$$\log|f(\alpha)| \le \int_{\tilde{X}_T} \log|f(z)| \, d\mu(z, w) = \int_T \log|f(z)| \, d\bar{\mu}(z).$$

So $\bar{\mu}$ is a Jensen measure for α with respect to $R(T)$. By assumption, $\bar{\mu}$ is the unit point mass at α. Hence

$$\mu(\Gamma_\alpha) = \int_{\Gamma_\alpha} d\mu = \int_{\{\alpha\}} d\bar{\mu} = 1.$$

Since μ is a probability measure, $\operatorname{supp} \mu \subseteq \Gamma_\alpha$.

Now $w \neq 0$ on $\tilde{X}_T$, so $1/w \in R(\tilde{X}_T)$. Thus

$$\left|\frac{1}{\beta}\right| = \left|\int_{\tilde{X}_T} \frac{1}{w} \, d\mu\right| \le \int_{\Gamma_\alpha} \frac{1}{|w|} \, d\mu = \frac{1}{(1 - |\alpha|^2)^{1/2}}$$

or $|\beta| \ge (1 - |\alpha|^2)^{1/2}$. Since $(\alpha, \beta) \in B$ we must have $|\beta| = (1 - |\alpha|^2)^{1/2}$, whence $(\alpha, \beta) \in \tilde{X}_T$.

Lemma 10. *Let T be a compact subset of the open disk such that $R(T) \neq C(T)$. Then $R(\tilde{X}_T) \neq C(\tilde{X}_T)$.*

Proof. Let $\mu \in C(T)^*$, $\mu \perp R(T)$, $\mu \neq 0$. If $z_0 \in \operatorname{int} \Delta$ let m_{z_0} be normalized Lebesgue measure on the circle $\Gamma_{z_0} = \{(z_0, w) \mid |w| = (1 - |z_0|^2)^{1/2}\}$. Define a measure $\tilde{\mu}$ on $\tilde{X}_T$ by

$$\int_{\tilde{X}_T} g\, d\bar{\mu} = \int_T \left[\int_{\Gamma_z} g(z, w)\, dm_z(w) \right] d\mu(z), \qquad g \in C(\tilde{X}_T).$$

Note that $\bar{\mu} \neq 0$. We wish to show that $\bar{\mu} \perp R(\tilde{X}_T)$, so let $f(z, w)$ be a rational function holomorphic in a neighborhood of $\tilde{X}_T$. For z near T in $\mathbf{C}$ we may define a function

$$a(z) = \int_{\Gamma_z} f(z, w)\, dm_z(w);$$

we assert that $a(z)$ is analytic in a neighborhood of T.

Fix $z_0 \in T$. For some $\epsilon > 0$ and $R_1 > (1 - |z_0|^2)^{1/2} > R_2 > 0$, $f(z, w)$ is analytic on

$$\Omega = \{(z, w) \in \mathbf{C}^2 |\ |z - z_0| < \epsilon,\ R_1 > |w| > R_2\}.$$

On Ω, $f(z, w)$ may be expanded in a Laurent series in w:

$$f(z, w) = \sum_{n=-\infty}^{\infty} a_n(z) w^n,$$

where

$$a_n(z) = \frac{1}{2\pi i} \int_{|w| = (1 - |z_0|^2)^{1/2}} \frac{f(z, w)\, dw}{w^{n+1}}$$

is analytic in z. But then

$$a(z) = \int_{\Gamma_z} f(z, w)\, dm_z(w) = \int_{\Gamma_z} \sum_{n=-\infty}^{\infty} a_n(z) w^n\, dm_z(w) = a_0(z)$$

is analytic for z near z_0. So the assertion is proven.

Consequently we have

$$\int_{\tilde{X}_T} f\, d\bar{\mu} = \int_T \left[\int_{\Gamma_z} f(z, w)\, dm_z(w) \right] d\mu(z) = \int_T a(z)\, d\mu(z) = 0.$$

Thus $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. This completes the proof of Theorem 3.

3. We wish to consider sets of the type $\tilde{X}_T$ described in §2, where we again require that T be a compact subset of $\operatorname{int} \Delta$. We impose the additional requirement that T be a "Swiss cheese", so that $R(T) \neq C(T)$. For our purposes a Swiss

cheese T is a compact subset of $\mathbf{C}$ with empty interior, formed by deleting from a piecewise smoothly bounded connected compact set S a sequence of nonempty piecewise smoothly bounded simply connected open sets O_m subject to the conditions:

(i) $\overline{O}_m \cap \overline{O}_n = \emptyset$ if $m \neq n$;

(ii) $\overline{O}_m \subseteq \operatorname{int} S$ for all m;

(iii) $\Sigma_{m=1}^{\infty}$ length $(\partial O_m) < \infty$.

With this definition the example in Browder's book [9, p. 192] is a Swiss cheese.

Theorem 4. *Let $T \subseteq \operatorname{int}\Delta$ be a Swiss cheese. Let K be a compact subset of $\widetilde{X}_T$ with nonempty interior in the topology of $\widetilde{X}_T$. Then $R(K) \neq C(K)$. Consequently, the restriction of $R(\widetilde{X}_T)$ to K is not dense in $C(K)$. (Note that, in particular, T may be chosen so that $\widetilde{X}_T$ satisfies Theorem* 3.)

Proof. We will construct a measure μ on K such that $\mu \perp R(K)$ and $\mu \neq 0$. For simplicity we may suppose that $(0, 1)$ is in the interior of K relative to $\widetilde{X}_T$. It follows that, for some positive $\epsilon < 1$ and some positive $\theta_0 < \pi$, K contains the set $F \cap \widetilde{X}_T$ where

$$F = \{(z, w) \in \partial B \mid |z| \leq \epsilon, \ |\arg w| \leq \theta_0\}.$$

Since T is a Swiss cheese, we may write $T = S \setminus \bigcup_{m=1}^{\infty} O_m$ with S and O_m as in the above definition. Let

$$K_n = \left\{(z, w) \in F \mid z \in S \setminus \bigcup_{m=1}^{n} O_m\right\}, \text{ if } n > 0;$$

so that $K_{n+1} \subseteq K_n$ for all $n > 0$, and $\bigcap_{n=1}^{\infty} K_n = K \cap F$. If U is any small open neighborhood of

$$\{(z, \theta) \in \mathbf{C} \times R \mid |z| \leq \epsilon, \ |\theta| \leq \theta_0\},$$

then the map $\psi\colon U \to \partial B$ defined by

$$\psi(z, \theta) = (z, e^{i\theta}(1 - |z|^2)^{1/2}), \qquad (z, \theta) \in U,$$

establishes a real analytic diffeomorphism of U with an open neighborhood of K_1 as a subset of ∂B. For $n = 1, 2, \cdots$ let $\partial_0 K_n$ denote the boundary of K_n relative to ∂B.

We may visualize $K_1, K_2, \cdots$ and $\partial_0 K_1, \partial_0 K_2, \cdots$ by means of the map ψ^{-1}:

$$\psi^{-1}(K_n) = \left\{ z \in S \setminus \bigcup_{m=1}^{n} O_m \,\middle|\, |z| \leq \epsilon \right\} \times \{|\theta| \leq \theta_0\}, \qquad n > 0;$$

$$\psi^{-1}(\partial_0 K_n) = \left\{ z \in \partial S \cup \left(\bigcup_{m=1}^{n} \partial O_m \right) \,\middle|\, |z| \leq \epsilon \right\} \times \{|\theta| \leq \theta_0\}$$

$$\cup \left\{ z \in S \setminus \bigcup_{m=1}^{n} O_m \,\middle|\, |z| = \epsilon \right\} \times \{|\theta| \leq \theta_0\}$$

$$\cup \left\{ z \in S \setminus \bigcup_{m=1}^{n} O_m \,\middle|\, |z| \leq \epsilon \right\} \times \{-\theta_0, \theta_0\}, \qquad n > 0.$$

From this it is evident that, for each n, $n = 1, 2, \cdots$, the set K_n is the closure of an open subset V_n of ∂B and

(i) $\partial_0 K_n$ is "piecewise smooth";

(ii) V_n "lies on one side of $\partial_0 K_n$".

We may therefore apply Stokes' theorem to V_n and $\partial_0 K_n$. (See [15, pp. 271–275] for precise definitions and a statement of an appropriate form of Stokes' theorem.)

Let f be analytic in a neighborhood of K_n. The preceding remarks show that

$$(1) \qquad \int_{\partial_0 K_n} f\, dz \wedge dw = \int_{V_n} \left(\frac{\partial f}{\partial \bar{z}} d\bar{z} + \frac{\partial f}{\partial \bar{w}} d\bar{w} \right) \wedge dz \wedge dw = 0.$$

Let μ_n denote the measure on $\partial_0 K_n$ corresponding to the form $dz \wedge dw$. We may regard $\{\mu_n\}$ as a sequence of measures on K_1, because $K_n \subseteq K_1$ for all $n > 0$. We assert that the condition

$$(2) \qquad \sum_{m=1}^{\infty} \text{length}\,(\partial O_m) < \infty$$

implies that the μ_n converge in norm. In fact, suppose that f is C^∞ in a neighborhood of K_1 and $\sup_{K_1} |f| \leq 1$. If $N > M > 0$, then

$$\left|\int_{K_1} f\,d\mu_N - \int_{K_1} f\,d\mu_M\right| = \left|\int_{\partial_0 K_N} f(z,w)\,dz\wedge dw - \int_{\partial_0 K_M} f(z,w)\,dz\wedge dw\right|$$

$$= \left|\int_{\psi^{-1}(\partial_0 K_N)} f(\psi(z,\theta))\,d(z\circ\psi)\wedge d(w\circ\psi) - \int_{\psi^{-1}(\partial_0 K_M)} f(\psi(z,\theta))\,d(z\circ\psi)\wedge d(w\circ\psi)\right|$$

$$= \left|\sum_{m=M+1}^{N}\left(\int_{\{(z,\theta)\mid |z|\le\epsilon,\, z\in\partial O_m,\, |\theta|\le\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge\left(\frac{\partial w}{\partial\bar z}d\bar z + \frac{\partial w}{\partial\theta}d\theta\right)\right.\right.$$

$$- \int_{\{(z,\theta)\mid |z|=\epsilon,\, z\in O_m,\, |\theta|\le\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge\left(\frac{\partial w}{\partial\bar z}d\bar z + \frac{\partial w}{\partial\theta}d\theta\right)$$

$$\left.\left.- \int_{\{(z,\theta)\mid |z|\le\epsilon,\, z\in O_m,\, \theta=\pm\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge\left(\frac{\partial w}{\partial\bar z}d\bar z + \frac{\partial w}{\partial\theta}d\theta\right)\right)\right|$$

$$\le \sum_{m=M+1}^{N}\left(\left|\int_{\{|z|\le\epsilon,\, z\in\partial O_m,\, |\theta|\le\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge(ie^{i\theta}(1-|z|^2)^{1/2}\,d\theta)\right|\right.$$

$$+\left|\int_{\{|z|=\epsilon,\, z\in O_m,\, |\theta|\le\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge(ie^{i\theta}(1-|z|^2)^{1/2}\,d\theta)\right|$$

$$\left.+\left|\int_{\{|z|\le\epsilon,\, z\in O_m,\, \theta=\pm\theta_0\}} f(z, e^{i\theta}(1-|z|^2)^{1/2})\,dz\wedge\left(\frac{-e^{i\theta}z}{2(1-|z|^2)^{1/2}}d\bar z\right)\right|\right)$$

$$\le \sum_{m=M+1}^{N}\left(\sup_{K_1}|f|\cdot\text{length}(\partial O_m)\cdot 2\theta_0 + \sup_{K_1}|f|\cdot\text{length}(\partial O_m)\cdot 2\theta_0\right.$$

$$\left.+ 2\sup_{K_1}|f|\cdot 2\,\text{area}(O_m)\cdot\frac{\epsilon}{2(1-\epsilon^2)^{1/2}}\right)$$

$$\le \sum_{m=M+1}^{N}\left(4\theta_0\,\text{length}(\partial O_m) + \frac{2\epsilon}{(1-\epsilon^2)^{1/2}}\text{area}(O_m)\right)$$

which by (2) converges to zero independent of f as $M, N\to\infty$.

Let μ be the uniform limit of the μ_n. Since $\operatorname{supp}\mu_n \subseteq K_n$ and $K_{n+1}\subseteq K_n$, we have $\operatorname{supp}\mu\subseteq\bigcap_{n=1}^{\infty}K_n = K\cap F$. If f is analytic in a neighborhood of K, then f is analytic in a neighborhood of some K_M, and for $N\ge M$ we have $\int_{\partial_0 K_N} f\,d\mu_N = 0$, by (1). So

$$\int_K f\,d\mu = \lim_{N\to\infty}\int_{\partial_0 K_N} f\,d\mu_N = 0.$$

Thus $\mu\perp R(K)$, and it remains only to show that $\mu\ne 0$.

Choose $M>0$ so that $O_M\cap\{|z|<\epsilon\}\ne\emptyset$. Let $g(z)$ be a bounded Borel function on $\mathbf{C}$ such that $g(z)=0$ unless $z\in\{|z|\le\epsilon\}\cap\partial O_M$, and such that

$$\int_{\{|z|\le\epsilon,\, z\in\partial O_M\}} g(z)\,dz = 1.$$

Let $f(z, w) = g(z)/iw$; then for $N \geq M$, μ_N agrees with μ_M on the support of f, so

$$\int_{K_1} f\,d\mu_N = \int_{K_1} f\,d\mu_M = \int_{\partial_0 K_M} \frac{g(z)}{iw} dz \wedge dw$$

$$= \int_{\{|z| \leq \epsilon,\, z \in \partial O_M,\, |\theta| \leq \theta_0\}} \frac{g(z)}{ie^{i\theta}(1-|z|^2)^{1/2}} dz \wedge (ie^{i\theta}(1 - |z|^2)^{1/2}\, d\theta)$$

$$= 2\theta_0 \int_{\{|z| \leq \epsilon,\, z \in \partial O_M\}} g(z)\, dz = 2\theta_0 \neq 0.$$

Thus $\int_K f\, d\mu \neq 0$, and so $\mu \neq 0$.

4. Our final objective is the construction of smooth sets with the pathological behavior discussed in the first two sections. The key result we need is that for any compact subset X of $\mathbf{C}^2$ we can find a smooth function f such that z, w, f generate $R(X)$. More precisely, we have the following result:

Lemma 11. *Let X be a compact subset of $\mathbf{C}^n$. There is an $f \in C_0^\infty(\mathbf{C}^n)$ such that the restrictions to X of $z_1, \cdots, z_n$ and f generate $R(X)$. Furthermore, f may be chosen so that $f|_{h_r(X)} \in R(h_r(X))$,* i.e., *$f|_{h_r(X)}$ coincides with the natural extension of $f|_X$ to the maximal ideal space of $R(X)$, $h_r(X)$.*

As the proof is simply a modification of the proof that $R(X)$ has $n+1$ generators (see, e.g., [9, p. 16]), we leave it to be included as an appendix.

One addition to the notation previously introduced is needed.

ρ_X, for X a compact subset of ∂B, will denote an arbitrarily chosen C^∞ real-valued function on ∂B which is zero precisely on X. It is easy to construct such a function. A similar function was used by Freeman in Example 5.3 in [16].

Lemma 12. *Let X be a compact subset of ∂B. Let $\mathfrak{A}$ be a uniform algebra on X which contains the polynomials in z and w, and suppose that there is a function $f \in C^\infty(\partial B)$ such that z, w, f generate $\mathfrak{A}$. Define a uniform algebra $\mathfrak{A}'$ on ∂B by*

$$\mathfrak{A}' = \{g \in C(\partial B) |\ g|_X \in \mathfrak{A}\}.$$

Let $\Phi: \partial B \to \mathbf{C}^6$ by

$$\Phi(z, w) = (z, w, f(z, w), \rho_X(z, w), \bar{z}\rho_X(z, w), \bar{w}\rho_X(z, w)).$$

Then Φ establishes an isometric isomorphism between the algebras $\mathfrak{A}'$ and $P[\Phi(\partial B)]$ by $F \leftrightarrow F \circ \Phi^{-1}$ for $F \in \mathfrak{A}'$.

Proof. First observe that z, w, f, ρ_X, $\bar{z}\rho_X$ and $\bar{w}\rho_X$ are in $\mathfrak{A}'$. In fact, we claim that they generate $\mathfrak{A}'$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}'$ generated by these functions, and let μ be any measure on ∂B orthogonal to $\mathfrak{B}$. We will show that $\mathfrak{A}' = \mathfrak{B}$ by showing that $\mu \perp \mathfrak{A}'$.

Now if $g \in C(\partial B)$ and $g|_X \equiv 0$, then g is uniformly approximable on ∂B by polynomials in $z, w, \rho_X, \bar{z}\rho_X, \bar{w}\rho_X$—this follows from the Stone-Weierstrass theorem for locally compact spaces, since the real-valued functions $\rho_X(z+\bar{z})$, $\rho_X(z-\bar{z})$, $\rho_X(w+\bar{w})$ and $\rho_X(w-\bar{w})$ separate the points of $\partial B \setminus X$. Thus $\operatorname{supp}\mu \subseteq X$. But $\mathfrak{B}$ contains all polynomials in z, w and f, so by the hypothesis that z, w and f generate $\mathfrak{A}$, we have $\mu \perp \mathfrak{A}'$.

Since the coordinates of the map Φ generate $\mathfrak{A}'$, it follows that Φ establishes the required isomorphism.

Theorem 5. *There is a C^∞ 3-sphere $\Sigma_1 \subseteq \mathbf{C}^6$ such that $h(\Sigma_1)\setminus\Sigma_1$ is nonempty but does not contain analytic structure.*

Proof. Let $Y = X_S$ with S chosen as in Wermer [7] or as in the example on p.365 with $b = 0$. Y has the following relevant properties:

(i) $(0, 0) \in h_r(Y)$;

(ii) $h_r(Y)$ does not contain analytic structure;

(iii) $Y \subseteq \partial\Delta^2$, the topological boundary of the unit bicylinder.

By Lemma 11 there is an $f \in C^\infty(\mathbf{C}^2)$ such that z, w and f generate $R(Y)$, and $f \in R(h_r(Y))$.

Let $X = h_r(Y) \cap \partial B$, $\mathfrak{A} =$ the uniform algebra on X generated by z, w, f.

We observe as follows that the maximal ideal space of $\mathfrak{A}$, which we shall denote by M, may be identified with $h_r(Y) \cap B$.

If $(z_0, w_0) \in h_r(Y) \cap B$, and G is a polynomial in z, w, f, then

$$|G(z_0, w_0)| \le \max\{|G(z,w)| \mid (z,w) \in [(\text{Silov boundary } R(Y)) \cap (B \cap h_r(Y))] \cup (\partial B \cap h_r(Y))\}$$

$$= \max_X |G|$$

by the Rossi local maximum modulus principle. Thus evaluation at (z_0, w_0) induces a homomorphism on $\mathfrak{A}$.

If, on the other hand, m is a homomorphism on $\mathfrak{A}$, then since z, w, f generate $R(Y)$, m induces a homomorphism of $R(Y)$ and so corresponds to evaluation of z, w, f at some point $(z_0, w_0) \in h_r(Y)$. Since B is polynomially convex, we must have $(z_0, w_0) \in h_r(Y) \cap B$.

We have therefore established the following facts about the maximal ideal space of $\mathfrak{A}$:

(i) $(0, 0) \in M$;

(ii) M does not contain analytic structure.

Let $\mathfrak{A}' = \{F \in C(\partial B) | \, F|_X \in \mathfrak{A}\}$. The maximal ideal space of $\mathfrak{A}'$ may evidently by identified with $\partial B \cup M$, and the Silov boundary of $\mathfrak{A}'$ is ∂B. Clearly the maximal ideal space of $\mathfrak{A}'$ does not contain analytic structure.

Let Φ be the map of $(z, w, f, \rho_X, \bar{z}\rho_X, \bar{w}\rho_X)$ as described in Lemma 12. Let $\Sigma_1 = \Phi(\partial B)$. By Lemma 12, $\mathfrak{A}'$ and $P(\Sigma_1)$ are equivalent uniform algebras, so $h(\Sigma_1)\backslash\Sigma_1$ is nonempty but Σ_1 does not contain analytic structure.

Note. The same technique may be applied to Stolzenberg's example [8] to obtain such a Σ_1 in $\mathbf{C}^5$ rather than $\mathbf{C}^6$.

Theorem 6. *There is a polynomially convex* C^∞ *3-sphere* $\Sigma_2 \subseteq \mathbf{C}^6$ *such that every point of* Σ_2 *is a peak point for* $P(\Sigma_2)$, *but* $P(\Sigma_2) \neq C(\Sigma_2)$.

Proof. Let X be the set $\tilde{X}_T$ of Theorem 3. Since $X \subseteq \partial B$, by Lemma 11 there is an $f \in C^\infty(\partial B)$ such that z, w and f generate $R(X)$. Let

$$\mathfrak{A} = R(X), \qquad \mathfrak{A}' = \{F \in C(\partial B) | \, F|_X \in \mathfrak{A}\}.$$

Clearly the maximal ideal space of $\mathfrak{A}'$ is ∂B, every point of ∂B is a peak point for $\mathfrak{A}'$, but $\mathfrak{A}' \neq C(\partial B)$. Let Φ be the corresponding map as described in Lemma 12, and let $\Sigma_2 = \Phi(\partial B)$. Then $\mathfrak{A}'$ and $P(\Sigma_2)$ are equivalent uniform algebras, so Σ_2 has the required properties.

APPENDIX (PROOF OF LEMMA 11)

Notation. If $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ with each α_i a nonnegative integer, then D^α is the differential operator on $C_0^\infty(\mathbf{C}^n)$ defined by

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n \partial^{\alpha_{n+1}} \bar{z}_1 \cdots \partial^{\alpha_{2n}} \bar{z}_n}, \qquad f \in C_0^\infty(\mathbf{C}^n),$$

where $|\alpha| = \sum_{i=1}^{2n} \alpha_i$.

Lemma 11. *Let* X *be a compact subset of* $\mathbf{C}^n$. *There is an* $f \in C_0^\infty(\mathbf{C}^n)$ *such that the restrictions to* X *of* $z_1, \cdots, z_n$ *and* f *generate* $R(X)$. *Furthermore,* f *may be chosen so that* $f|_{h_r(X)} \in R(h_r(X))$, *i.e.,* $f|_{h_r(X)}$ *coincides with the natural extension of* $f|_X$ *to* $h_r(X)$.

Proof of lemma. Choose a sequence of polynomials $g_1, g_2, \cdots$ so that each g_m has no zero on X, and so that $\{p/g_m | \, p$ is a polynomial, $1 \leq m < \infty\}$ is dense in $R(X)$. For each m let h_m be a C_0^∞-extension of $1/g_m$ from $h_r(X)$ to $\mathbf{C}^n$ (note that g_m does not vanish on $h_r(X)$), and arrange so that the supports of the h_m's are all contained in some fixed compact subset of $\mathbf{C}^n$.

Now choose positive constants $c_1, c_2, \cdots$ so that

(i) $c_m \sup_{x\in \mathbf{C}^n} \sup_{0\le|\alpha|\le m} |D^\alpha h_m(x)| < 2^{-m}$,

(ii) for $1 \le k < m$, $c_m \sup_{x\in X} |g_k(x)/g_m(x)| < 2^{-m}c_k$, for $m = 1, 2, \cdots$. Let $f = \sum_{k=1}^\infty c_k h_k$. By (i), $f \in C_0^\infty(\mathbf{C}^n)$, and $f|_{h_r(X)} \in R(h_r(X))$.

The proof may now be concluded exactly as in [9]; for this purpose we henceforth regard all functions as functions on X, so that for example $h_k = 1/g_k$. We will show that the algebra A generated by $z_1, z_2, \cdots, z_n, f$ is $R(X)$. By choice of $\{g_m\}$, is it sufficient to show that $1/g_k \in A$ for all k.

Let $f_k = \sum_{m=k}^\infty c_m/g_m$, $k = 1, 2, \cdots$. Then $f_1 = f \in A$, and if $1/g_1, 1/g_2, \cdots, 1/g_{k-1} \in A$, then $f_k \in A$. Proceed by induction: if $f_k \in A$, then $f_k g_k \in A$, and $f_k g_k = c_k + \sum_{m=k+1}^\infty c_m g_k/g_m$. By (ii), $c_k > \sup_X |\sum_{m=k+1}^\infty c_m g_k/g_m|$, so $(f_k g_k)^{-1} \in A$, so $1/g_k \in A$. Thus the induction is complete and so is the proof of the lemma.

BIBLIOGRAPHY

1. J. Wermer, *Polynomial approximation on an arc in* $\mathbf{C}^3$, Ann. of Math. (2) 62 (1955), 269–270. MR 17, 255.

2. W. Rudin, *Subalgebras of spaces of continuous functions*, Proc. Amer. Math. Soc. 7 (1956), 825–830. MR 18, 587.

3. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

4. K. de Leeuw, *A type of convexity in the space of n complex variables*, Trans. Amer. Math. Soc. 83 (1956), 193–204. MR 18, 294.

5. ———, *Functions on circular subsets of the space of n complex variables*, Duke Math. J. 24 (1957), 415–431. MR 20 #1235.

6. H. Rossi, *The local maximum modulus principle*, Ann. of Math. (2) 72 (1960), 1–11. MR 22 #8317.

7. J. Wermer, *On an example of Stolzenberg*, Sympos. Several Complex Variables (Park City, Utah, 1970), Lecture Notes in Math., no. 184, Springer-Verlag, Berlin and New York, 1971, pp. 79–84.

8. G. Stolzenberg, *A hull with no analytic structure*, J. Math. Mech. 12 (1963), 103–111. MR 26 #627.

9. A Browder, *Introduction to function algebras*, Benjamin, New York, 1969. MR 39 #7431.

10. B. J. Cole, *One-point parts and the peak point conjecture*, Ph.D. Dissertation, Yale University, New Haven, Conn., 1968.

11. R. McKissick, *A non-trivial normal sup norm algebra*, Bull. Amer. Math. Soc. 69 (1963), 391–395. MR 26 #4166.

12. A Huber, *Über Potentiale, welche auf vorgegebenen Mengen verschwinden*, Comment. Math. Helv. 43 (1968), 41–50. MR 37 #450.

13. A Browder, *Point derivations on function algebras*, J. Functional Analysis 1 (1967), 22–27. MR 35 #2144.

14. M. S. Mel'nikov, *On the structure of Gleason parts of the algebra* $R(E)$, Funkcional. Anal. i Priložen. 1 (1967), 97–100. (Russian) MR 35 #740.

15. W. Fleming, *Functions of several variables*, Addison-Wesley, Reading, Mass., 1965. MR 30 #4875.

16. M. Freeman, *Some conditions on uniform approximation on a manifold*, Function Algebras, Proc. Internat. Sympos. on Function Algebras (Tulane University, 1965), Scott, Foresman, Chicago, Ill., 1965, pp. 42–60. MR 33 #1758.

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